

Smoothed Functional Algorithms for Stochastic Optimization using q -Gaussian Distributions

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Abstract

The importance of the q -Gaussian family of distributions lies in its power-law nature, and its close association with Gaussian, Cauchy and uniform distributions. This class of distributions arises from maximization of a generalized information measure. We use the power-law nature of the q -Gaussian distribution to improve upon the smoothing properties of Gaussian and Cauchy distributions. Based on this, we propose a Smoothed Functional (SF) scheme for gradient estimation using q -Gaussian distribution. Our work extends the class of distributions that can be used in SF algorithms by including the q -Gaussian distributions, which encompass the above three distributions as special cases. Using the derived gradient estimates, we propose two-timescale algorithms for optimization of a stochastic objective function with gradient descent method. We prove that the proposed algorithms converge to a local optimum. Performance of the algorithms is shown by simulation results on a queuing model.

1 Introduction

Shannon (1948) provided the concept of entropy as a measure of information, or more precisely, a measure of uncertainty given by probability distributions. Rényi (1961) was the first to introduce the concept of generalized measures of information when he proposed the first well-known generalization of Shannon entropy, known as α -entropy or Rényi entropy, based on Kolmogorov-Nagumo averages. Several other generalizations have also been studied in the literature (Perez, 1968; Daróczy, 1970), and have been extensively used in physics, communication theory and other disciplines.

One of the most recently studied generalized information measure is the nonextensive entropy, due to Tsallis (1988), defined as

$$H_q(p) = - \sum_{x \in \mathcal{X}} p(x)^q \ln_q(p(x)), \quad (1)$$

where p is the probability mass function of the discrete random variable on the set \mathcal{X} , and the q -logarithm is defined as $\ln_q(x) = \frac{x^{1-q} - 1}{1-q}$, $q \in \mathbb{R}$, $q \neq 1$. It is called nonextensive because of its pseudo-additive nature (Tsallis, 1988). Although this generalization had been introduced earlier in (Havrda and Charvát, 1967), Tsallis provided interpretations in the context of statistical mechanics. Suyari (2004) generalized the Shannon-Khinchin axioms to the nonextensive case.

The most important characteristic of these generalized information measures is that, on maximization, they give rise to power-law distributions, while maximization of Shannon entropy gives rise to exponential distributions. The concept of maximization of information measure can be attributed to Kullback's minimum discrimination theorem (Kullback, 1959), which establishes important connections between statistics and information theory. This theorem shows that exponential distributions can be obtained by minimizing Kullback-Leibler divergence under given moment constraints. One can consider maximum entropy (Jaynes, 1957) as a special case, where maximization of Shannon entropy under moment constraints leads to exponential distributions. For example, given mean and variance of a random variable, maximum entropy gives rise to Gaussian distribution.

While exponential distributions have been extensively studied and used in statistical modeling, the power-law behavior has been observed in most of the real-world data, e.g., (Barabási and Albert, 1999;

Pareto, 1906). The importance of Tsallis entropy can be attributed to its connections with power-law distributions, which has been studied in different contexts like finance, earthquakes and network traffic (Abe and Suzuki, 2003, 2005). Compared to the exponential family, the Tsallis distributions, *i.e.*, the family of distributions resulting from maximization of Tsallis entropy, have an additional shape parameter q , similar to that in (1) that controls the nature of the power-law tails. One of the most studied Tsallis entropy maximizer is the q -Gaussian distribution (Prato and Tsallis, 1999), which is a power-law generalization of Gaussian distribution. In this paper, we study the q -Gaussian distribution in the context of smoothed functional algorithms for stochastic optimization.

Stochastic techniques play a key role in optimization problems, where the objective function does not have an analytic expression. Such problems are often encountered in discrete event systems, which are quite common in engineering and financial world. Most often, the data, obtained via statistical survey or simulation, contains only noisy estimates of the objective function to be optimized. One of the most commonly used solution methodologies involves stochastic approximation algorithms, originally due to Robbins and Monro (1951), which is used to find the zeros of a given function. Based on this approach, gradient descent algorithms have been developed, in which the parameters controlling the system track the zeros of the gradient of the objective. However, these algorithms require an estimate of the cost gradient. Kiefer and Wolfowitz (1952) provide such a gradient estimate using several parallel simulations of the system. More efficient techniques for gradient estimation, have been developed based on the smoothed functional approach (Katkovnik and Kulchitsky, 1972; Bhatnagar and Borkar, 2003), simultaneous perturbation stochastic approximation (Spall, 1992) etc. A stochastic variation of Newton-based optimization methods, also known as adaptive Newton-based schemes has also been studied in literature (Ruppert, 1985; Spall, 2000; Bhatnagar, 2007).

When the above schemes for gradient estimation are employed in optimization methods involving long-run average cost objective, the time complexity of the algorithms increase as the long-run average cost needs to be estimated after each parameter update. A more efficient approach is to simultaneously perform the long-run averaging and parameter updates using different step-size schedules. These classes of algorithms constitute the multi-timescale stochastic approximation algorithms (Bhatnagar and Borkar, 1998). Two-timescale optimization algorithms have been developed using simultaneous perturbations (Bhatnagar et al., 2003) and smoothed functional (Bhatnagar, 2007) schemes. The main issue with such algorithms is that, although convergence of the algorithm to a local optimum is guaranteed, the global optimum is often not achieved in practice. Bhatnagar (2007) proves that the gradient SF schemes based on Gaussian perturbations converge to a local minimum, and provides comparison of the performance of various multi-timescale algorithms for stochastic optimization on a queuing system. The results presented there indicate that the performance of the SF algorithms depends considerably on several tuning parameters, such as the variance of the Gaussian distribution, and also the step-sizes.

Summary of our contributions

The smoothed functional schemes for simulation based optimization have become popular due to their smoothing effects on local fluctuations. We derive for the first time, smoothed functional algorithms with power-law (q -Gaussian) perturbations.

Prior work (Rubinstein, 1981; Styblinski and Tang, 1990) indicated that the class of distributions that can be used for the perturbation random variables in SF algorithms includes Gaussian, Cauchy and uniform distributions. Our main contribution is to show that the q -Gaussian family of distributions belong to this class, and encompasses the above three distributions as special cases. This allows us to work with a larger class of distributions in SF algorithms, in an unified way, where the “shape parameter” of the q -Gaussian controls its power-law behavior. This parameter also controls the smoothness of the convolution, thereby providing additional tuning.

We show that the multivariate q -Gaussian distribution satisfies all the conditions for smoothing kernels discussed in (Rubinstein, 1981). We then present estimators for gradient of a function using the q -Gaussian smoothing kernel. We also present multi-timescale algorithms for stochastic optimization using q -Gaussian based SF that incorporate gradient based search procedures, and prove the convergence of the proposed algorithms to the neighborhood of a local optimum. The convergence analysis presented in this paper differs from the approaches that have been studied earlier (Bhatnagar, 2007). Here, we provide a more straightforward technique using standard results from (Borkar, 2008; Kushner and Clark, 1978). Further, we perform simulations on a queuing network to illustrate the benefits of the q -Gaussian based SF algorithms compared to their Gaussian counterparts. A shorter version of this paper containing only the one-simulation q -Gaussian SF algorithm, and without the convergence proof, has been presented

in IEEE 2012 International Symposium on Information Theory (Ghoshdastidar et al., 2012).

The rest of the paper is organized as follows. The framework for the optimization problem and some preliminaries on SF and q -Gaussians are presented in Section 2. Section 3 validates the use of q -Gaussian as smoothing kernel, and presents gradient descent algorithms using q -Gaussian SF. The convergence analysis of the proposed algorithms is discussed in Section 4. Section 5 presents simulations based on a numerical setting. Finally, Section 6 provides the concluding remarks. In the appendix, we discuss a sampling technique for multivariate q -Gaussians that is used in the proposed algorithms.

2 Background and Preliminaries

2.1 Problem Framework

Let $\{Y_n : n \in \mathbb{N}\} \subset \mathbb{R}^d$ be a parameterized Markov process, depending on a tunable parameter $\theta \in C$, where C is a compact and convex subset of \mathbb{R}^N . Let $P_\theta(x, dy)$ denote the transition kernel of $\{Y_n\}$ when the operative parameter is $\theta \in C$. Let $h : \mathbb{R}^d \mapsto \mathbb{R}^+ \cup \{0\}$ be a Lipschitz continuous cost function associated with the process.

Assumption I. *The process $\{Y_n\}$ is ergodic for any given θ as the operative parameter, i.e., as $L \rightarrow \infty$,*

$$\frac{1}{L} \sum_{m=0}^{L-1} h(Y_m) \rightarrow \mathbb{E}_{\nu_\theta}[h(Y)],$$

where ν_θ is the stationary distribution of $\{Y_n\}$.

Our objective is to minimize the long-run average cost

$$J(\theta) = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{m=0}^{L-1} h(Y_m) = \int_{\mathbb{R}^d} h(x) \nu_\theta(dx), \quad (2)$$

by choosing an appropriate $\theta \in C$. The existence of the above limit is assured by Assumption I and the fact that h is continuous, hence measurable. In addition, we assume that the average cost $J(\theta)$ satisfies the following requirement.

Assumption II. *The function $J(\cdot)$ is continuously differentiable for all $\theta \in C$.*

Definition 2.1 (Non-anticipative sequence). *A random sequence of parameter vectors, $(\theta(n))_{n \geq 0} \subset C$, controlling a process $\{Y_n\} \subset \mathbb{R}^d$, is said to be non-anticipative if the conditional probability $P(Y_{n+1} \in dy | \mathcal{F}_n) = P_\theta(Y_n, dy)$ almost surely for $n \geq 0$ and all Borel sets $dy \subset \mathbb{R}^d$, where $\mathcal{F}_n = \sigma(\theta(m), Y_m, m \leq n)$, $n \geq 0$ are associated σ -fields.*

One can verify that under a non-anticipative parameter sequence $(\theta(n))$, the sequence $(Y_n, \theta(n))_{n \geq 0}$ is Markov. We assume the existence of a stochastic Lyapunov function.

Assumption III. *Let $(\theta(n))$ be a non-anticipative sequence of random parameters controlling the process $\{Y_n\}$, and $\mathcal{F}_n = \sigma(\theta(m), Y_m, m \leq n)$, $n \geq 0$ be a sequence of associated σ -fields. There exists $\epsilon_0 > 0$, a compact set $K \subset \mathbb{R}^d$, and a continuous function $V : \mathbb{R}^d \mapsto \mathbb{R}^+ \cup \{0\}$, with $\lim_{\|x\| \rightarrow \infty} V(x) = \infty$, such that*

- (i) $\sup_n \mathbb{E}[V(Y_n)^2] < \infty$, and
- (ii) $\mathbb{E}[V(Y_{n+1}) | \mathcal{F}_n] \leq V(Y_n) - \epsilon_0$, whenever $Y_n \notin K$, $n \geq 0$.

While Assumption II is a technical requirement, Assumption III ensures that the process under a tunable parameter remains stable. Assumption III will not be required, for instance, if, in addition, the single-stage cost function h is bounded. It can be seen that the sequence of parameters obtained using any of our algorithms below form a non-anticipative sequence.

2.2 Smoothed Functionals

Here, we present the idea behind the smoothed functional approach proposed by Katkovnik and Kulchitsky (1972). We consider a real-valued function $f : C \mapsto \mathbb{R}$, defined over a compact set C . Its smoothed functional is defined as

$$S_\beta[f(\theta)] = \int_{-\infty}^{\infty} G_\beta(\eta) f(\theta - \eta) d\eta = \int_{-\infty}^{\infty} G_\beta(\theta - \eta) f(\eta) d\eta, \quad (3)$$

where $G_\beta : \mathbb{R}^N \mapsto \mathbb{R}$ is a kernel function, with a parameter β taking values from \mathbb{R} . The idea behind using smoothed functionals is that if $f(\theta)$ is not well-behaved, *i.e.*, it has a fluctuating character, then $S_\beta[f(\theta)]$ is “better-behaved”. This can ensure that any optimization algorithm with objective function $f(\theta)$ does not get stuck at a local minimum, but converges to a global minimum. The parameter β controls the degree of smoothness. Rubinstein (1981) established that the SF algorithm achieves these properties if the kernel function satisfies the following sufficient conditions:

(P1) $G_\beta(\eta) = \frac{1}{\beta^N} G\left(\frac{\eta}{\beta}\right)$, where $G(x)$ corresponds to $G_\beta(x)$ with $\beta = 1$, *i.e.*,

$$G\left(\frac{\eta}{\beta}\right) = G_1\left(\frac{\eta^{(1)}}{\beta}, \frac{\eta^{(2)}}{\beta}, \dots, \frac{\eta^{(N)}}{\beta}\right),$$

(P2) $G_\beta(\eta)$ is piecewise differentiable in η ,

(P3) $G_\beta(\eta)$ is a probability distribution function, *i.e.*, $S_\beta[f(\theta)] = \mathbb{E}_{G_\beta(\eta)}[f(\theta - \eta)]$,

(P4) $\lim_{\beta \rightarrow 0} G_\beta(\eta) = \delta(\eta)$, where $\delta(\eta)$ is the Dirac delta function, and

(P5) $\lim_{\beta \rightarrow 0} S_\beta[f(\theta)] = f(\theta)$.

A two-sided form of SF is defined as

$$\begin{aligned} S'_\beta[f(\theta)] &= \frac{1}{2} \int_{-\infty}^{\infty} G_\beta(\eta) (f(\theta - \eta) + f(\theta + \eta)) d\eta \\ &= \frac{1}{2} \int_{-\infty}^{\infty} G_\beta(\theta - \eta) f(\eta) d\eta + \frac{1}{2} \int_{-\infty}^{\infty} G_\beta(\eta - \theta) f(\eta) d\eta. \end{aligned} \quad (4)$$

The Gaussian distribution satisfies the above conditions, and has been used as a smoothing kernel (Katkovnik and Kulchitsky, 1972; Styblinski and Tang, 1990). The SF approach provides a method for estimating the gradient of any function, which satisfies Assumptions I–III (Bhatnagar and Borkar, 2003). Bhatnagar (2007) uses the Gaussian smoothing, and derives a gradient estimator from (3) as

$$\nabla_\theta J(\theta) \approx \frac{1}{\beta ML} \sum_{n=0}^{M-1} \sum_{m=0}^{L-1} \eta(n) h(Y_m) \quad (5)$$

for large M , L and small β . The stochastic process $\{Y_m\}$ is governed by parameter $(\theta(n) + \beta\eta(n))$, where $\theta(n) \in C \subset \mathbb{R}^N$ is obtained through an iterative scheme, and $\eta(n) = (\eta^{(1)}(n), \dots, \eta^{(N)}(n))^T$ is a N -dimensional vector of i.i.d. standard Gaussian random variable. Similarly, a two-simulation gradient estimator has been suggested using (4), which is of the following form

$$\nabla_\theta J(\theta) \approx \frac{1}{2\beta ML} \sum_{n=0}^{M-1} \sum_{m=0}^{L-1} \eta(n) (h(Y_m) - h(Y'_m)) \quad (6)$$

for large M , L and small β , where $\{Y_m\}$ and $\{Y'_m\}$ are two processes governed by parameters $(\theta(n) + \beta\eta(n))$ and $(\theta(n) - \beta\eta(n))$, respectively, $\theta(n)$ and $\eta(n)$ being as before.

2.3 Generalized information measure and the q -Gaussian distribution

A continuous form of the Shannon entropy, also known as differential entropy, has been extensively studied in statistical mechanics, probability and statistics. It is defined as

$$H(p) = \int_{\mathcal{X}} p(x) \ln p(x) \, dx, \quad (7)$$

where $p(\cdot)$ is a p.d.f. defined on the sample space \mathcal{X} . Following the lines of discrete form generalization given in (Tsallis, 1988), Dukupati et al. (2007) provides a measure theoretic formulation of continuous form Tsallis entropy functional, defined as

$$H_q(p) = \frac{1 - \int_{\mathcal{X}} (p(x))^q \, dx}{q - 1}, \quad q \in \mathbb{R}, q \neq 1. \quad (8)$$

This function results when the natural logarithm in (7) is replaced by the q -logarithm defined earlier. The differential Shannon entropy can be retrieved from (8) as $q \rightarrow 1$.

The q -Gaussian distribution was developed to describe the process of Lévy super-diffusion (Prato and Tsallis, 1999), but has been later studied in other fields, such as finance (Sato, 2010) and statistics (Suyari, 2005). Its importance lies in its power-law nature, due to which the tails of the q -Gaussian decay at a slower rate than the Gaussian distribution, depending on q . It results from maximizing Tsallis entropy under certain ‘deformed’ moment constraints, known as normalized q -expectation defined by

$$\langle f \rangle_q = \frac{\int_{\mathbb{R}} f(x) p(x)^q \, dx}{\int_{\mathbb{R}} p(x)^q \, dx}. \quad (9)$$

This form of an expectation considers an escort distribution $p_q(x) = \frac{p(x)^q}{\int_{\mathbb{R}} p(x)^q \, dx}$, and has been shown to be compatible with the foundations of nonextensive statistics (Tsallis et al., 1998). Prato and Tsallis (1999) maximized Tsallis entropy under the constraints, $\langle x \rangle_q = \mu_q$ and $\langle (x - \mu)^2 \rangle_q = \beta_q^2$, which are known as q -mean and q -variance, respectively. These are generalizations of standard first and second moments, and tend to the usual mean and variance, respectively, as $q \rightarrow 1$. This results in the q -Gaussian distribution¹ that has the form

$$G_{q,\beta_q}(x) = \frac{1}{\beta_q K_q} \left(1 - \frac{(1-q)}{(3-q)\beta_q^2} (x - \mu_q)^2 \right)_+^{\frac{1}{1-q}} \quad \text{for all } x \in \mathbb{R}, \quad (10)$$

where, $y_+ = \max(y, 0)$ is called the Tsallis cut-off condition (Tsallis, 1995), which ensures that the above expression is defined, and K_q is the normalizing constant, which is given by

$$K_q = \begin{cases} \frac{\sqrt{\pi}\sqrt{3-q}}{\sqrt{1-q}} \frac{\Gamma(\frac{2-q}{1-q})}{\Gamma(\frac{5-3q}{2(1-q)})} & \text{for } -\infty < q < 1, \\ \frac{\sqrt{\pi}\sqrt{3-q}}{\sqrt{q-1}} \frac{\Gamma(\frac{3-q}{2(q-1)})}{\Gamma(\frac{1}{q-1})} & \text{for } 1 < q < 3, \end{cases}$$

with Γ being the Gamma function, which exists over the specified intervals. The function defined in (10) is not integrable for $q \geq 3$, and hence, q -Gaussian is a probability density function only when $q < 3$.

A multivariate form of the q -Gaussian distribution has been discussed in (Umarov and Tsallis, 2007; Vignat and Plastino, 2007). Considering the q -mean and q -covariance matrix to be μ_q and Σ_q respectively, the N -variate q -Gaussian distribution can be expressed as

$$G_{q,\Sigma_q}(X) = \frac{1}{K_{q,N} |\Sigma_q|^{1/2}} \left(1 - \frac{(1-q)}{(N+2-Nq)} (X - \mu_q)^T \Sigma_q^{-1} (X - \mu_q) \right)_+^{\frac{1}{1-q}} \quad (11)$$

for all $X \in \mathbb{R}^N$, where the normalizing constant

$$K_{q,N} = \begin{cases} \left(\frac{N+2-Nq}{1-q} \right)^{\frac{N}{2}} \frac{\pi^{N/2} \Gamma(\frac{2-q}{1-q})}{\Gamma(\frac{2-q}{1-q} + \frac{N}{2})} & \text{for } q < 1, \\ \left(\frac{N+2-Nq}{q-1} \right)^{\frac{N}{2}} \frac{\pi^{N/2} \Gamma(\frac{1}{q-1} - \frac{N}{2})}{\Gamma(\frac{1}{q-1})} & \text{for } 1 < q < \left(1 + \frac{2}{N} \right). \end{cases} \quad (12)$$

¹ The notation G_{q,β_q} is used to maintain consistency with smoothed functional notations, which is used in Section 3.

As in the one-dimensional case, the distribution is only defined for $q < 1 + \frac{2}{N}$. can be mapped to the multivariate Students'- t distribution. The support set of the above distribution is given by

$$\Omega_q = \begin{cases} \left\{ x \in \mathbb{R}^N : (X - \mu_q)^T \Sigma_q^{-1} (X - \mu_q) < \frac{N+2-Nq}{1-q} \right\} & \text{for } q < 1, \\ \mathbb{R}^N & \text{for } 1 < q < \left(1 + \frac{2}{N}\right). \end{cases} \quad (13)$$

The multivariate normal distribution can be obtained as a special case when $q \rightarrow 1$, while for $q \rightarrow -\infty$, we obtain the uniform distribution with an infinitesimally small support around its mean. It is interesting to note that for $q > 1$, these distributions have a one-to-one correspondence with Student's- t distribution, and in particular, for $q = (1 + \frac{2}{N+1})$, we retrieve the Cauchy distribution. A similar distribution can also be obtained by maximizing Rényi entropy (Costa et al., 2003). In this paper, we study the multivariate q -Gaussian distribution as a smoothing kernel, and develop smoothed functional algorithms based on it.

3 q -Gaussian based Smoothed Functional Algorithms

3.1 q -Gaussian as a Smoothing Kernel

The first step in applying q -Gaussians for SF algorithms is to ensure that the distribution satisfies the Rubinstein conditions (properties (P1)–(P5) in Section 2.2). The rest of the paper uses the multivariate form of q -Gaussian (11), with the q -mean $\mu_q = 0$, and q -covariance matrix $\Sigma_q = \beta^2 I_{N \times N}$, i.e., the components are uncorrelated.

Proposition 3.1. *N -dimensional q -Gaussian distribution (11), with q -covariance $\beta^2 I_{N \times N}$ satisfies the kernel properties (P1)–(P5) for all $q < (1 + \frac{2}{N})$, $q \neq 1$.*

Proof. (P1) From (11), it is evident that $G_{q,\beta}(x) = \frac{1}{\beta^N} G_q\left(\frac{x}{\beta}\right)$.

(P2) For $1 < q < (1 + \frac{2}{N})$, $G_{q,\beta}(x) > 0$ for all $x \in \mathbb{R}^N$. Thus,

$$\nabla_x G_{q,\beta}(x) = -\frac{2x}{(N+2-Nq)\beta^2} \frac{G_{q,\beta}(x)}{\left(1 - \frac{(1-q)}{(N+2-Nq)\beta^2} \|x\|^2\right)}. \quad (14)$$

For $q < 1$, (14) holds when $x \in \Omega_q$. On the other hand, when $x \notin \Omega_q$, we have $G_{q,\beta}(x) = 0$ and hence, $\nabla_x G_{q,\beta}(x) = 0$. Thus, $G_{q,\beta}(x)$ is differentiable for $q > 1$, and piecewise differentiable for $q < 1$.

(P3) $G_{q,\beta}(x)$ is a distribution for $q < (1 + \frac{2}{N})$ and hence, the corresponding SF $S_{q,\beta}(\cdot)$, parameterized by both q and β , can be written as $S_{q,\beta}[f(\theta)] = \mathbb{E}_{G_{q,\beta}(x)}[f(\theta - x)]$.

(P4) $G_{q,\beta}$ is a distribution satisfying $\lim_{\beta \rightarrow 0} G_{q,\beta}(0) = \infty$. So, $\lim_{\beta \rightarrow 0} G_{q,\beta}(x) = \delta(x)$.

(P5) This property trivially holds due to convergence in mean as

$$\lim_{\beta \rightarrow 0} S_{q,\beta}[f(\theta)] = \int_{-\infty}^{\infty} \lim_{\beta \rightarrow 0} G_{q,\beta}(x) f(\theta - x) dx = \int_{-\infty}^{\infty} \delta(x) f(\theta - x) dx = f(\theta).$$

Hence the claim. \square

From the above result, it follows that q -Gaussian can be used as a kernel function, and hence, given a particular value $q \in (-\infty, 1) \cup (1, 1 + \frac{2}{N})$ and some $\beta > 0$, the one-sided and two-sided SFs of any function $f : \mathbb{R}^N \mapsto \mathbb{R}$ are respectively given by

$$S_{q,\beta}[f(\theta)] = \int_{\Omega_q} G_{q,\beta}(\theta - x) f(x) dx, \quad (15)$$

$$S'_{q,\beta}[f(\theta)] = \frac{1}{2} \int_{\Omega_q} G_{q,\beta}(\theta - x) f(x) dx + \frac{1}{2} \int_{\Omega_q} G_{q,\beta}(x - \theta) f(x) dx, \quad (16)$$

where the nature of the SFs are controlled by both q and β .

3.2 One-simulation q -Gaussian SF Gradient Estimate

The objective is to estimate the gradient of the average cost $\nabla_\theta J(\theta)$ using the SF approach, where existence of $\nabla_\theta J(\theta)$ follows from Assumption II. The gradient of smoothed functional (smoothed gradient) is defined as (Rubinstein, 1981)

$$\nabla_\theta S_{q,\beta}[J(\theta)] = \int_{\Omega_q} \nabla_\theta G_{q,\beta}(\theta - \eta) J(\eta) d\eta ,$$

where Ω_q is the support set defined as in (13). As there is no functional relationship between θ and η over Ω_q , *i.e.*, $\frac{d\eta^{(j)}}{d\theta^{(i)}} = 0$ for all i, j ,

$$\begin{aligned} \nabla_\theta^{(i)} G_{q,\beta}(\theta - \eta) &= \frac{1}{\beta^N K_{q,N}} \frac{2(\eta^{(i)} - \theta^{(i)})}{\beta^2(N+2-Nq)} \left(1 - \frac{(1-q) \sum_{k=1}^N (\theta^{(k)} - \eta^{(k)})^2}{(N+2-Nq)\beta^2} \right)^{\frac{q}{1-q}} \\ &= \frac{2}{\beta^2(N+2-Nq)} \frac{(\eta^{(i)} - \theta^{(i)})}{\rho(\frac{\theta-\eta}{\beta})} G_{q,\beta}(\theta - \eta) , \end{aligned} \quad (17)$$

where $\rho(\eta) = \left(1 - \frac{(1-q)}{(N+2-Nq)} \|\eta\|^2 \right)$. Hence, substituting $\eta' = \frac{\eta-\theta}{\beta}$, and using the symmetry of $G_{q,\beta}(\cdot)$ and $\rho(\cdot)$, we can write

$$\begin{aligned} \nabla_\theta S_{q,\beta}[J(\theta)] &= \left(\frac{2}{\beta(N+2-Nq)} \right) \int_{\Omega_q} \frac{\eta'}{\rho(\eta')} G_q(\eta') J(\theta + \beta\eta') d\eta' \\ &= \left(\frac{2}{\beta(N+2-Nq)} \right) \mathbb{E}_{G_q(\eta')} \left[\frac{\eta'}{\rho(\eta')} J(\theta + \beta\eta') \right] . \end{aligned} \quad (18)$$

In the sequel (Proposition 4.8), we show that $\|\nabla_\theta S_{q,\beta}[J(\theta)] - \nabla_\theta J(\theta)\| \rightarrow 0$ as $\beta \rightarrow 0$. Hence, for large M and small β , the form of gradient estimate suggested by (18) is

$$\nabla_\theta J(\theta) \approx \left(\frac{2}{\beta(N+2-Nq)M} \right) \sum_{n=0}^{M-1} \left(\frac{\eta(n) J(\theta + \beta\eta(n))}{\rho(\eta(n))} \right) , \quad (19)$$

where $\eta(1), \eta(2), \dots, \eta(n)$ are uncorrelated identically distributed standard q -Gaussian distributed random vectors. Considering that in two-timescale algorithms (discussed later), the value of θ is updated concurrently with the gradient estimation procedure, we estimate $\nabla_\theta J(\theta(n))$ at each stage. By ergodicity assumption (Assumption I), we can write (19) as

$$\nabla_\theta J(\theta(n)) \approx \left(\frac{2}{\beta M L (N+2-Nq)} \right) \sum_{n=0}^{M-1} \sum_{m=0}^{L-1} \frac{\eta(n) h(Y_m)}{\left(1 - \frac{(1-q)}{(N+2-Nq)} \|\eta(n)\|^2 \right)} \quad (20)$$

for large L , where the process $\{Y_m\}$ has the same transition kernel as defined in Assumption I, except that it is governed by parameter $(\theta(n) + \beta\eta(n))$.

3.3 Two-simulation q -Gaussian SF Gradient Estimate

In a similar manner, based on (4), the gradient of the two-sided SF can be written as

$$\nabla_\theta S'_{q,\beta}[J(\theta)] = \frac{1}{2} \int_{\Omega_q} \nabla_\theta G_\beta(\theta - \eta) J(\eta) d\eta + \frac{1}{2} \int_{\Omega_q} \nabla_\theta G_\beta(\eta - \theta) J(\eta) d\eta . \quad (21)$$

The first integral can be obtained as in (18). The second integral is evaluated as

$$\int_{\Omega_q} \nabla_\theta G_\beta(\eta - \theta) J(\eta) d\eta = \frac{2}{\beta(N+2-Nq)} \int_{\Omega_q} \frac{\eta'}{\rho(\eta')} G_q(\eta') J(\theta - \beta\eta') d\eta' ,$$

where $\eta' = \frac{\theta - \eta}{\beta}$. Thus, we obtain the gradient as a conditional expectation

$$\nabla_{\theta} S'_{q,\beta}[J(\theta)] = \left(\frac{1}{\beta(N+2-Nq)} \right) \mathbb{E}_{G_q(\eta)} \left[\frac{\eta}{\rho(\eta)} \left(J(\theta + \beta\eta) - J(\theta - \beta\eta) \right) \middle| \theta \right]. \quad (22)$$

In sequel (Proposition 4.10) we show that $\left\| \nabla_{\theta} S'_{q,\beta}[J(\theta)] - \nabla_{\theta} J(\theta) \right\| \rightarrow 0$ as $\beta \rightarrow 0$, which can be used to approximate (22), for large M , L and small β , as

$$\nabla_{\theta} J(\theta(n)) \approx \frac{1}{\beta M L (N+2-Nq)} \sum_{n=0}^{M-1} \sum_{m=0}^{L-1} \frac{\eta(n)(h(Y_m) - h(Y'_m))}{\left(1 - \frac{(1-q)}{(N+2-Nq)} \|\eta(n)\|^2 \right)} \quad (23)$$

where $\{Y_m\}$ and $\{Y'_m\}$ are governed by $(\theta(n) + \beta\eta(n))$ and $(\theta(n) - \beta\eta(n))$ respectively.

3.4 Proposed Gradient Descent Algorithms

We propose two-timescale algorithms based on the estimates obtained in (20) and (23). Let $(a(n))_{n \geq 0}$ and $(b(n))_{n \geq 0}$ be two step-size sequences satisfying the following.

Assumption IV. $(a(n))_{n \geq 0}$ and $(b(n))_{n \geq 0}$ are two positive step-size sequences satisfying $\sum_{n=0}^{\infty} a(n)^2 < \infty$,

$\sum_{n=0}^{\infty} b(n)^2 < \infty$, $\sum_{n=0}^{\infty} a(n) = \sum_{n=0}^{\infty} b(n) = \infty$ and $a(n) = o(b(n))$, i.e., $\frac{a(n)}{b(n)} \rightarrow 0$ as $n \rightarrow \infty$.

It must be noted that in the algorithms, although M is chosen to be a large quantity (to ensure convergence), the quantity L is arbitrarily picked and can be any finite positive number. The averaging of the inner summation in (20) and (23) is obtained in our algorithms using two-timescale stochastic approximation. In principle, one may select $L = 1$. However, it is generally observed that a value of L typically between 5 and 500 results in better performance (Bhatnagar, 2007). Further, the algorithms require generation of N -dimensional random vectors, consisting of uncorrelated q -Gaussian distributed random variates. This method is described in Appendix.

For $\theta = (\theta^{(1)}, \dots, \theta^{(N)})^T \in \mathbb{R}^N$, let $\mathcal{P}_C(\theta) = (\mathcal{P}_C(\theta^{(1)}), \dots, \mathcal{P}_C(\theta^{(N)}))^T$ represent the projection of θ onto the set C . For simulation, we need to project the perturbed random vectors $(\theta(n) + \beta\eta(n))$ onto C using the above projection. The quantities $(Z^{(i)}(n), i = 1, \dots, N)_{n \geq 0}$ are used to estimate $\nabla_{\theta} J(\theta)$ in the recursions.

The Gq-SF1 Algorithm

- 1 Fix M , L , q and β ;
 - 2 Set $Z^{(i)}(0) = 0, i = 1, \dots, N$;
 - 3 Fix the parameter vector $\theta(0) = (\theta^{(1)}(0), \theta^{(2)}(0), \dots, \theta^{(N)}(0))^T$;
 - 4 **for** $n = 0$ **to** $M - 1$ **do**
 - 5 Generate a random vector $\eta(n) = (\eta^{(1)}(n), \eta^{(2)}(n), \dots, \eta^{(N)}(n))^T$ from a standard N -dimensional q -Gaussian distribution;
 - 6 **for** $m = 0$ **to** $L - 1$ **do**
 - 7 Generate the simulation Y_{nL+m} governed with parameter $\mathcal{P}_C(\theta(n) + \beta\eta(n))$;
 - 8 **for** $i = 1$ **to** N **do**
 - 9 $Z^{(i)}(nL + m + 1) = (1 - b(n))Z^{(i)}(nL + m) + b(n) \left[\frac{2\eta^{(i)}(n)h(Y_{nL+m})}{\beta(N+2-Nq)\left(1 - \frac{(1-q)}{(N+2-Nq)} \|\eta(n)\|^2\right)} \right]$;
 - 10 **end**
 - 11 **end**
 - 12 **for** $i = 1$ **to** N **do**
 - 13 $\theta^{(i)}(n+1) = \mathcal{P}_C \left(\theta^{(i)}(n) - a(n)Z^{(i)}(nL) \right)$;
 - 14 **end**
 - 15 Set $\theta(n+1) = (\theta^{(1)}(n+1), \theta^{(2)}(n+1), \dots, \theta^{(N)}(n+1))^T$;
 - 16 **end**
 - 17 Output $\theta(M) = (\theta^{(1)}(M), \dots, \theta^{(N)}(M))^T$ as the final parameter vector;
-

The Gq-SF2 algorithm is similar to the Gq-SF1 algorithm, except that we use two parallel simulations Y_{nL+m} and Y'_{nL+m} governed with parameters $(\theta(n) + \beta\eta(n))$ and $(\theta(n) - \beta\eta(n))$ respectively, and update the gradient estimate, in Step 9, using the single-stage cost function of both simulations as in (23).

The Gq-SF2 Algorithm

```

1 Fix  $M, L, q$  and  $\beta$ ;
2 Set  $Z^{(i)}(0) = 0, i = 1, \dots, N$ ;
3 Fix the parameter vector  $\theta(0) = (\theta^{(1)}(0), \theta^{(2)}(0), \dots, \theta^{(N)}(0))^T$ ;
4 for  $n = 0$  to  $M - 1$  do
5   Generate a random vector  $\eta(n) = (\eta^{(1)}(n), \eta^{(2)}(n), \dots, \eta^{(N)}(n))^T$  from a standard  $N$ -dimensional
    $q$ -Gaussian distribution;
6   for  $m = 0$  to  $L - 1$  do
7     Generate two simulations  $Y_{nL+m}$  and  $Y'_{nL+m}$  governed with control parameters  $\mathcal{P}_C(\theta(n) + \beta\eta(n))$  and
      $\mathcal{P}_C(\theta(n) - \beta\eta(n))$  respectively;
8     for  $i = 1$  to  $N$  do
9        $Z^{(i)}(nL + m + 1) = (1 - b(n))Z^{(i)}(nL + m) + b(n) \left[ \frac{\eta^{(i)}(n)(h(Y_{nL+m}) - h(Y'_{nL+m}))}{\beta(N+2-Nq)\left(1 - \frac{(1-q)}{(N+2-Nq)}\|\eta(n)\|^2\right)} \right]$ ;
10    end
11  end
12  for  $i = 1$  to  $N$  do
13     $\theta^{(i)}(n+1) = \mathcal{P}_C\left(\theta^{(i)}(n) - a(n)Z^{(i)}(nL)\right)$ ;
14  end
15  Set  $\theta(n+1) = (\theta^{(1)}(n+1), \theta^{(2)}(n+1), \dots, \theta^{(N)}(n+1))^T$ ;
16 end
17 Output  $\theta(M) = (\theta^{(1)}(M), \dots, \theta^{(N)}(M))^T$  as the final parameter vector;
```

4 Convergence of the proposed Algorithms

We provide here a more straightforward technique to prove that the algorithms converge to a local optimum as compared to (Bhatnagar, 2007). Before presenting the details of convergence analysis, we present the following result on q -Gaussians. It provides an expression for the moments of N -variate q -Gaussian distributed random vector. This is a consequence of the results presented in (Gradshteyn and Ryzhik, 1994). This result plays a key role in the proofs discussed below.

Proposition 4.1. *Suppose $X = (X^{(1)}, X^{(2)}, \dots, X^{(N)}) \in \mathbb{R}^N$ is a random vector, where the components are uncorrelated and identically distributed, each being distributed according to a q -Gaussian distribution with zero q -mean and unit q -variance, with parameter $q \in (-\infty, 1) \cup (1, 1 + \frac{2}{N})$. Also, let $\rho(X) = \left(1 - \frac{(1-q)}{(N+2-Nq)}\|X\|^2\right)$. Then, for any $b, b_1, b_2, \dots, b_N \in \mathbb{Z}^+ \cup \{0\}$, we have*

$$\mathbb{E}_{G_q} \left[\frac{(X^{(1)})^{b_1} (X^{(2)})^{b_2} \dots (X^{(N)})^{b_N}}{(\rho(X))^b} \right] = \begin{cases} \bar{K} \left(\frac{N+2-Nq}{|1-q|} \right)^{\sum_{i=1}^N \frac{b_i}{2}} \left(\prod_{i=1}^N \frac{b_i!}{2^{b_i} (\frac{b_i}{2})!} \right) & \text{if } b_i \text{ is even for all } i = 1, 2, \dots, N, \\ 0 & \text{otherwise,} \end{cases} \quad (24)$$

where

$$\bar{K} = \begin{cases} \frac{\Gamma(\frac{1}{1-q} - b + 1) \Gamma(\frac{1}{1-q} + 1 + \frac{N}{2})}{\Gamma(\frac{1}{1-q} + 1) \Gamma(\frac{1}{1-q} - b + 1 + \frac{N}{2} + \sum_{i=1}^N \frac{b_i}{2})} & \text{if } q \in (-\infty, 1), \\ \frac{\Gamma(\frac{1}{q-1}) \Gamma(\frac{1}{q-1} + b - \frac{N}{2} - \sum_{i=1}^N \frac{b_i}{2})}{\Gamma(\frac{1}{q-1} + b) \Gamma(\frac{1}{q-1} - \frac{N}{2})} & \text{if } q \in (1, 1 + \frac{2}{N}), \end{cases} \quad (25)$$

exists only if the arguments in the above Gamma functions are positive, which holds for $b < \left(1 + \frac{1}{1-q}\right)$ if $q < 1$, and $\left(\frac{1}{q-1} - \frac{N}{2}\right) > \left(\sum_{i=1}^N \frac{b_i}{2} - b\right)$ if $1 < q < \left(1 + \frac{2}{N}\right)$.

Proof. Since $\Sigma_q = I_{N \times N}$, and $\rho(X)$ is non-negative over Ω_q , we have

$$\begin{aligned} \mathbb{E}_{G_q(X)} \left[\frac{(X^{(1)})^{b_1} (X^{(2)})^{b_2} \dots (X^{(N)})^{b_N}}{(\rho(X))^b} \right] \\ = \frac{1}{K_{q,N}} \int_{\Omega_q} (x^{(1)})^{b_1} (x^{(2)})^{b_2} \dots (x^{(N)})^{b_N} \left(1 - \frac{(1-q)}{(N+2-Nq)} \|x\|^2 \right)^{\frac{1}{1-q}-b} dx. \end{aligned}$$

The second equality in (24) can be easily proved. If for some $i = 1, \dots, N$, b_i is odd, then the above function is odd, and its integration is zero over Ω_q , which is symmetric with respect to any axis by definition. For the other cases, since the function is even, the integral is same over every orthant. Hence, we may consider the integration over the first orthant, *i.e.*, where each component is positive. For $q < 1$, we can reduce the above integral, using (Gradshteyn and Ryzhik, 1994, equation (4.635)), to obtain

$$\mathbb{E}_{G_q(X)} \left[\frac{\prod_{i=1}^N (X^{(i)})^{b_i}}{(\rho(X))^b} \right] = \frac{\prod_{i=1}^N \Gamma\left(\frac{b_i+1}{2}\right)}{K_{q,N} \Gamma(\bar{b})} \left(\frac{N+2-Nq}{1-q} \right)^{\bar{b}} \int_0^1 (1-y)^{\left(\frac{1}{1-q}-b\right)} y^{(\bar{b}-1)} dy \quad (26)$$

where we set $\bar{b} = \left(\frac{N}{2} + \sum_{i=1}^N \frac{b_i}{2} \right)$. One can observe that the integral in (26) is in the form of a Beta function. Since b_i 's are even, we can expand $\Gamma\left(\frac{b_i+1}{2}\right)$ using the expansion of Gamma function of half-integers to get $\Gamma\left(\frac{b_i+1}{2}\right) = \frac{b_i!}{2^{b_i} \left(\frac{b_i}{2}\right)!} \sqrt{\pi}$. The claim can be obtained by substituting $K_{q,N}$ from (12) and using the relation $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. It is easy to verify that all the Gamma functions in the equality are positive provided $b < \left(1 + \frac{1}{1-q}\right)$. The result for the interval $1 < q < \left(1 + \frac{2}{N}\right)$ can be proved in a similar way (see equations (4.635) and (4.636) of (Gradshteyn and Ryzhik, 1994)). However, in this case the Gamma functions are positive if $b, b_1, b_2, \dots, b_N \in \mathbb{Z}^+ \cup \{0\}$ satisfy the mentioned condition. It may be noted here that this is always true for any dimension if $b > \sum_{i=1}^N \frac{b_i}{2}$ since q -Gaussians are defined only when $\frac{1}{q-1} > \frac{N}{2}$. \square

It is easy to verify the following result in the limiting case of $q \rightarrow 1$. This would help to ensure that the convergence analysis done in this paper also holds in the case of Gaussian SF.

Corollary 4.2. *In the limiting case, as $q \rightarrow 1$,*

$$\lim_{q \rightarrow 1} \mathbb{E}_{G_q(X)} \left[\frac{(X^{(1)})^{b_1} (X^{(2)})^{b_2} \dots (X^{(N)})^{b_N}}{(\rho(X))^b} \right] = \prod_{i=1}^N \mathbb{E}_{G(X)} \left[(X^{(i)})^{b_i} \right].$$

4.1 Convergence of Gq-SF1 Algorithm

First, let us consider the update along the faster timescale, *i.e.*, Step 9 of the Gq-SF1 algorithm. We define $\tilde{\theta}(p) = \theta(n)$, $\tilde{\eta}(p) = \eta(n)$ and $\tilde{b}(p) = b(n)$ for $nL \leq p < (n+1)L$, $n \geq 0$. It follows from Assumption IV that $a(p) = o(\tilde{b}(p))$, $\sum_p \tilde{b}(p) = \infty$ and $\sum_p \tilde{b}(p)^2 < \infty$. We can rewrite Step 9 of Algorithm 1 as the following iteration for all $p \geq 0$

$$Z(p+1) = Z(p) + \tilde{b}(p) [g(Y_p) - Z(p)], \quad (27)$$

where

$$g(Y_p) = \frac{2\tilde{\eta}(p)h(Y_p)}{\beta(N+2-Nq)\rho(\tilde{\eta}(p))}.$$

Here, $\rho(\cdot)$ is defined as in (17), and $\{Y_p : p \in \mathbb{N}\}$ is a Markov process parameterized by $\mathcal{P}_C(\tilde{\theta}(p) + \beta\tilde{\eta}(p))$. Let $\mathcal{G}_p = \sigma(\tilde{\theta}(k), \tilde{\eta}(k), Y_k, k \leq p)$ denote the σ -field generated by the mentioned quantities. We can observe that $(\mathcal{G}_p)_{p \geq 0}$ is a filtration, where $g(Y_p)$ is \mathcal{G}_p -measurable for each $p \geq 0$.

We summarize the results presented in (Borkar, 2008, Chapter 6, Lemma 3 – Theorem 9) in the following theorem. This result leads to the stability and convergence of iteration (27), which runs on the faster timescale.

Theorem 4.3. *Consider the iteration, $x_{p+1} = x_p + \gamma(p)[f(x_p, Y_p) + M_p]$. Let the following conditions hold:*

1. $\{Y_p : p \in \mathbb{N}\}$ is a Markov process satisfying Assumptions I and III,
2. for each $x \in \mathbb{R}^N$ and $x_p \equiv x$ for all $p \in \mathbb{N}$, Y_p has a unique invariant probability measure ν_x ,
3. $(\gamma(p))_{p \geq 0}$ are step-sizes satisfying $\sum_{p=0}^{\infty} \gamma(p) = \infty$ and $\sum_{p=0}^{\infty} \gamma^2(p) < \infty$,
4. $f(\cdot, \cdot)$ is Lipschitz continuous in its first argument uniformly w.r.t the second,
5. M_p is a martingale difference noise term with bounded variance,
6. if $\tilde{f}(x, \nu_x) = \mathbb{E}_{\nu_x}[f(x, Y)]$, then the limit $\hat{f}(x(t)) = \lim_{a \uparrow \infty} \frac{\tilde{f}(ax(t), \nu_{ax(t)})}{a}$ exists uniformly on compacts, and
7. the ODE $\dot{x}(t) = \hat{f}(x(t))$ is well-posed and has the origin as the unique globally asymptotically stable equilibrium.

Then the update x_p satisfies $\sup_p \|x_p\| < \infty$, almost surely, and converges to the stable fixed points of the ordinary differential equation (ODE)

$$\dot{x}(t) = \tilde{f}(x(t), \nu_{x(t)}).$$

Rewriting the update (27) as

$$Z(p+1) = Z(p) + \tilde{b}(p)[E[g(Y_p)|\mathcal{G}_{p-1}] - Z(p) + A_p], \quad (28)$$

where $A_p = g(Y_p) - E[g(Y_p)|\mathcal{G}_{p-1}]$ is \mathcal{G}_p -measurable. The following result shows that $(A_p, \mathcal{G}_p)_{p \geq 0}$ satisfies Condition 5 in Theorem 4.3.

Lemma 4.4. *For all values of $q \in (-\infty, 1) \cup (1, 1 + \frac{2}{N})$, $(A_p, \mathcal{G}_p)_{p \in \mathbb{N}}$ is a martingale difference sequence with a bounded variance.*

Proof. It is easy to see that for all $p \geq 0$, $E[A_p|\mathcal{G}_{p-1}] = 0$. So $(A_p, \mathcal{G}_p)_{p \in \mathbb{N}}$ is a martingale difference sequence. Expanding the terms, we have

$$\begin{aligned} & \mathbb{E}[\|A_p\|^2 | \mathcal{G}_{p-1}] \\ & \leq \frac{8}{\beta^2(N+2-Nq)^2} \mathbb{E} \left[\left(\frac{\|\tilde{\eta}(p)\| h(Y_p)}{\rho(\tilde{\eta}(p))} \right)^2 + \left(\mathbb{E} \left[\frac{\|\tilde{\eta}(p)\| h(Y_p)}{\rho(\tilde{\eta}(p))} \middle| \mathcal{G}_{p-1} \right] \right)^2 \middle| \mathcal{G}_{p-1} \right]. \end{aligned}$$

Applying conditional Jensen's inequality on the second term, we obtain

$$\mathbb{E}[\|A_p\|^2 | \mathcal{G}_{p-1}] \leq \frac{16}{\beta^2(N+2-Nq)^2} \mathbb{E} \left[\frac{\|\tilde{\eta}(p)\|^2}{\rho(\tilde{\eta}(p))^2} h^2(Y_p) \middle| \mathcal{G}_{p-1} \right]. \quad (29)$$

For $q \in (-\infty, 1)$, we use Holder's inequality to write (29) as

$$\begin{aligned} \mathbb{E}[\|A_p\|^2 | \mathcal{G}_{p-1}] & \leq \frac{16}{\beta^2(N+2-Nq)^2} \sup_{\eta} \left(\frac{\|\tilde{\eta}(p)\|^2}{\rho(\tilde{\eta}(p))^2} \right) \mathbb{E}[h^2(Y_p) | \mathcal{G}_{p-1}] \\ & \leq \frac{16}{\beta^2(1-q)(N+2-Nq)} \mathbb{E}[h^2(Y_p) | \mathcal{G}_{p-1}]. \end{aligned}$$

since, $\|\eta\|^2 < \frac{N+2-Nq}{1-q}$ and $\rho(\eta) \geq 1$ for all $\eta \in \Omega_q$. By Lipschitz continuity of h , there exists $\alpha_1 > 0$ such that $|h(Y_p)| \leq \alpha_1(1 + \|Y_p\|)$ for all p , and hence, by Assumption III, we can claim

$$\mathbb{E}[h(Y_p)^2 | \mathcal{G}_{p-1}] \leq 2\alpha_1^2 (1 + \mathbb{E}[\|Y_p\|^2 | \mathcal{G}_{p-1}]) < \infty \text{ a.s.} \quad (30)$$

On the other hand, for $q \in (1, 1 + \frac{2}{N})$, we apply Cauchy-Schwartz inequality for each of the components in (29) to obtain

$$\begin{aligned} \mathbb{E}[\|A_p\|^2 | \mathcal{G}_{p-1}] & \leq \frac{16}{\beta^2(N+2-Nq)^2} \sum_{j=1}^N \mathbb{E} \left[\frac{(\tilde{\eta}^{(j)}(p))^2}{\rho(\tilde{\eta}(p))^2} h^2(Y_p) \middle| \mathcal{G}_{p-1} \right] \\ & \leq \frac{16}{\beta^2(N+2-Nq)^2} \sum_{j=1}^N \mathbb{E} \left[\frac{(\tilde{\eta}^{(j)}(p))^4}{\rho(\tilde{\eta}(p))^4} \right]^{1/2} \mathbb{E}[h^4(Y_p) | \mathcal{G}_{p-1}]^{1/2}. \end{aligned}$$

The second expectation can be shown to be finite as in (30), while we apply Proposition 4.1 to study the existence of $\mathbb{E} \left[\frac{(\eta^{(j)})^4}{\rho(\eta)^4} \right]$. We can observe that in this case, $b = 4$ and $b_i = 4$ if $i = j$, otherwise $b_i = 0$. and so $b > \sum_{i=1}^N \frac{b_i}{2}$. Proposition 4.1 ensures that the term is finite, and hence, the claim. \square

We can write the parameter update (Step 13 of Gq-SF1) as

$$\theta^{(i)}(n+1) = \mathcal{P}_C \left(\theta^{(i)}(n) - \tilde{b}(n)\zeta(n) \right),$$

where $\zeta(n) = \frac{a(n)}{\tilde{b}(n)} Z^{(i)}(nL) = o(1)$ since $a(n) = o(\tilde{b}(n))$. Thus, the parameter update recursion can be seen to track the ODE

$$\dot{\theta}(t) = 0. \quad (31)$$

Hence, the recursion $\theta(n)$, $n \geq 0$ appears quasi-static when viewed from the timescale of $(\tilde{b}(n))$, and hence, in the update (28), one may let $\tilde{\theta}(p) \equiv \theta$ and $\tilde{\eta}(p) \equiv \eta$ for all $p \in \mathbb{N}$. Consider the following ODE

$$\dot{Z}(t) = \frac{2\eta J(\theta + \beta\eta)}{\beta(N+2-Nq)\rho(\eta)} - Z(t). \quad (32)$$

Lemma 4.5. *The sequence $(Z(p))$ is uniformly bounded with probability 1. Further,*

$$\left\| Z(p) - \left(\frac{2\tilde{\eta}(p)J(\tilde{\theta}(p) + \beta\tilde{\eta}(p))}{\beta(N+2-Nq)\rho(\tilde{\eta}(p))} \right) \right\| \rightarrow 0$$

almost surely as $p \rightarrow \infty$.

Proof. It can be easily verified that iteration (28) satisfies all the conditions of Theorem 4.3. Thus, by Theorem 4.3, $(Z(p))$ converges to ODE (32) as

$$\mathbb{E}_{\nu_{(\theta+\beta\eta)}} \left[\frac{2\eta h(Y_p)}{\beta(N+2-Nq)\rho(\eta)} \right] = \frac{2\eta J(\theta + \beta\eta)}{\beta(N+2-Nq)\rho(\eta)}.$$

We can also see that

$$\lim_{a \uparrow \infty} \frac{1}{a} \left(\frac{2\eta J(\theta + \beta\eta)}{\beta(N+2-Nq)\rho(\eta)} - aZ(t) \right) = -Z(t).$$

All the conditions in Theorem 4.3 are seen to be verified and the claim follows. \square

From Lemma 4.5, Steps 13 and 15 of Gq-SF1 can be written as

$$\begin{aligned} \theta(n+1) &= \mathcal{P}_C \left(\theta(n) - a(n) \left[\frac{2\eta(n)J(\theta(n) + \beta\eta(n))}{\beta(N+2-Nq)\rho(\eta(n))} \right] \right) \\ &= \mathcal{P}_C \left(\theta(n) + a(n) [-\nabla_{\theta(n)} J(\theta(n)) + \Delta(\theta(n)) + \xi_n] \right), \end{aligned} \quad (33)$$

where the error in the gradient estimate is given by

$$\Delta(\theta(n)) = \nabla_{\theta(n)} J(\theta(n)) - \nabla_{\theta(n)} S_{q,\beta} [J(\theta(n))] \quad (34)$$

and the noise term is

$$\begin{aligned} \xi_n &= \nabla_{\theta(n)} S_{q,\beta} [J(\theta(n))] - \frac{2\eta(n)J(\theta(n) + \beta\eta(n))}{\beta(N+2-Nq)\rho(\eta(n))} \\ &= \frac{2}{\beta(N+2-Nq)} \left(\mathbb{E}_{G_q(\eta)} \left[\frac{\eta(n)}{\rho(\eta(n))} J(\theta(n) + \beta\eta(n)) \middle| \theta(n) \right] - \frac{\eta(n)}{\rho(\eta(n))} J(\theta(n) + \beta\eta(n)) \right), \end{aligned} \quad (35)$$

which is a martingale difference term. Let $\mathcal{F}_n = \sigma(\theta(0), \dots, \theta(n), \eta(0), \dots, \eta(n-1))$ denote the σ -field generated by the mentioned quantities. We can observe that $(\mathcal{F}_n)_{n \geq 0}$ is a filtration, where ξ_0, \dots, ξ_{n-1} are \mathcal{F}_n -measurable for each $n \geq 0$.

We state the following result due Kushner and Clark (1978, Theorem 5.3.1, pp 189–196), adapted to our scenario, which leads to the convergence of the updates in (33).

Lemma 4.6. *Given the iteration, $x_{n+1} = \mathcal{P}_C(x_n + \gamma(n)(f(x_n) + \xi_n))$, where*

1. \mathcal{P}_C represents a projection operator onto a closed and bounded constraint set C ,
2. $f(\cdot)$ is a continuous function,
3. $(\gamma(n))_{n \geq 0}$ is a positive sequence satisfying $\gamma(n) \downarrow 0$, $\sum_{n=0}^{\infty} \gamma(n) = \infty$, and
4. $\sum_{n=0}^m \gamma(n) \xi_n$ converges a.s.

Under the above conditions, the update (x_n) converges to the asymptotically stable fixed points of the ODE

$$\dot{x}(t) = \tilde{\mathcal{P}}_C(f(x(t))), \quad (36)$$

$$\text{where } \tilde{\mathcal{P}}_C(f(x)) = \lim_{\epsilon \downarrow 0} \left(\frac{\mathcal{P}_C(x + \epsilon f(x)) - x}{\epsilon} \right).$$

The next result shows that the noise term ξ_n satisfies the last condition in Lemma 4.6, while the subsequent result proves the error term $\Delta(\theta(n))$ is considerably small.

Lemma 4.7. *Let $M_n = \sum_{k=0}^{n-1} a(k) \xi_k$. Then, for all values of $q \in (-\infty, 1) \cup (1, 1 + \frac{2}{N})$, $(M_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is an almost surely convergent martingale sequence.*

Proof. We can easily observe that for all $k \geq 0$, $\mathbb{E}[\xi_k | \mathcal{F}_k]$

$$= \frac{2}{\beta(N+2-Nq)} \left(\mathbb{E} \left[\frac{\eta(k)J(\theta(k) + \beta\eta(k))}{\beta\rho(\eta(k))} \middle| \theta(k) \right] - \mathbb{E} \left[\frac{\eta(k)J(\theta(k) + \beta\eta(k))}{\beta\rho(\eta(k))} \middle| \mathcal{F}_k \right] \right).$$

So $\mathbb{E}[\xi_k | \mathcal{F}_k] = 0$, since $\theta(k)$ is \mathcal{F}_k -measurable, whereas $\eta(k)$ is independent of \mathcal{F}_k . It follows that $(\xi_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is a martingale difference sequence, and hence $(M_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is a martingale sequence. Now, use of conditional Jensen's inequality leads to

$$\begin{aligned} \mathbb{E}[\|\xi_k\|^2 | \mathcal{F}_k] &= \sum_{j=1}^N \mathbb{E} \left[\left(\xi_k^{(j)} \right)^2 \middle| \mathcal{F}_k \right] \\ &\leq \frac{16}{\beta^2(N+2-Nq)^2} \sum_{j=1}^N \mathbb{E} \left[\frac{(\eta(k)^{(j)})^2}{\rho(\eta(k))^2} J(\theta(k) + \beta\eta(k))^2 \middle| \theta(k) \right]. \end{aligned}$$

For any $\eta \in \mathbb{R}^N$, by definition $J(\theta(k) + \beta\eta) = \mathbb{E}[h(Y_p)]$, where the expectation is with respect to the stationary measure. By Jensen's inequality, we can claim $J(\theta(k) + \beta\eta)^2 \leq \mathbb{E}[h(Y_p)^2]$ and $J(\theta(k) + \beta\eta)^4 \leq \mathbb{E}[h(Y_p)^4]$ for all $\eta \in \mathbb{R}^N$. Using these facts along with arguments similar to Lemma 4.4, it can be seen that $\sup_k \mathbb{E}[\|\xi_k\|^2 | \mathcal{F}_k] < \infty$ for all k , and hence, if $\sum_n a(n)^2 < \infty$,

$$\sum_{n=0}^{\infty} \mathbb{E}[\|M_{n+1} - M_n\|^2] = \sum_{n=0}^{\infty} a(n)^2 \mathbb{E}[\|\xi_n\|^2] \leq \sum_{n=0}^{\infty} a(n)^2 \sup_n \mathbb{E}[\|\xi_n\|^2] < \infty \text{ a.s.}$$

The claim follows from martingale convergence theorem (Williams, 1991, page 111). \square

Proposition 4.8. *For a given $q < (1 + \frac{2}{N})$, $q \neq 1$, and for all $\theta \in C$, the error term*

$$\left\| \nabla_{\theta} S_{q,\beta}[J(\theta)] - \nabla_{\theta} J(\theta) \right\| = o(\beta).$$

Proof. For small $\beta > 0$, using Taylor series expansion of $J(\theta + \beta\eta)$ around $\theta \in C$,

$$J(\theta + \beta\eta) = J(\theta) + \beta\eta^T \nabla_{\theta} J(\theta) + \frac{\beta^2}{2} \eta^T \nabla_{\theta}^2 J(\theta) \eta + o(\beta^2).$$

So we can write (18) as

$$\begin{aligned} \nabla_{\theta} S_{q,\beta}[J(\theta)] &= \frac{2}{(N+2-Nq)} \left(\frac{J(\theta)}{\beta} \mathbb{E}_{G_q(\eta)} \left[\frac{\eta}{\rho(\eta)} \right] + \mathbb{E}_{G_q(\eta)} \left[\frac{\eta\eta^T}{\rho(\eta)} \right] \nabla_{\theta} J(\theta) \right. \\ &\quad \left. + \frac{\beta}{2} \mathbb{E}_{G_q(\eta)} \left[\frac{\eta\eta^T \nabla_{\theta}^2 J(\theta) \eta}{\rho(\eta)} \middle| \theta \right] + o(\beta) \right). \end{aligned} \quad (37)$$

We consider each term in (37). The i^{th} component in the first term is $\mathbb{E}_{G_q(\eta)} \left[\frac{\eta^{(i)}}{\rho(\eta)} \right] = 0$ by Proposition 4.1 for all $i = 1, \dots, N$. Similarly, the i^{th} component in the third term can be written as

$$\frac{\beta}{2} \mathbb{E}_{G_q(\eta)} \left[\frac{\eta \eta^T \nabla_{\theta}^2 J(\theta) \eta}{\rho(\eta)} \right]^{(i)} = \frac{\beta}{2} \sum_{j=1}^N \sum_{k=1}^N [\nabla_{\theta}^2 J(\theta)]_{j,k} \mathbb{E}_{G_q(\eta)} \left[\frac{\eta^{(i)} \eta^{(j)} \eta^{(k)}}{\rho(\eta)} \right].$$

It can be observed that in all cases, each term in the summation is an odd function, and so from Proposition 4.1, we can show that the third term in (37) is zero. Using a similar argument, we claim that the off-diagonal terms in $\mathbb{E}_{G_q(\eta)} \left[\frac{\eta \eta^T}{\rho(\eta)} \right]$ are zero, while the diagonal terms are of the form $\mathbb{E}_{G_q(\eta)} \left[\frac{(\eta^{(i)})^2}{\rho(\eta)} \right]$, which exists for all $q \in (-\infty, 1) \cup (1, 1 + \frac{2}{N})$ as the conditions in Proposition 4.1 are always satisfied on this interval. Further,

$$\mathbb{E}_{G_q(\eta)} \left[\frac{(\eta^{(i)})^2}{\rho(\eta)} \right] = \frac{(N + 2 - Nq)}{2}. \quad (38)$$

The claim follows by substituting the above expression in (37). \square

Now, we consider the following ODE for the slowest timescale recursion

$$\dot{\theta}(t) = \tilde{\mathcal{P}}_C \left(-\nabla_{\theta} J(\theta(t)) \right), \quad (39)$$

where $\tilde{\mathcal{P}}_C(f(x)) = \lim_{\epsilon \downarrow 0} \left(\frac{\mathcal{P}_C(x + \epsilon f(x)) - x}{\epsilon} \right)$. In accordance with Lemma 4.6, it can be observed that the stable points of (39) lie in the set

$$K = \left\{ \theta \in C \mid \tilde{\mathcal{P}}_C \left(-\nabla_{\theta} J(\theta) \right) = 0 \right\}. \quad (40)$$

We have the following key result which shows that iteration (33) tracks ODE (39), and hence, the convergence of our algorithm is proved.

Theorem 4.9. *Under Assumptions I – IV, given $\epsilon > 0$ and $q \in (-\infty, 1) \cup (1, 1 + \frac{2}{N})$, there exists $\beta_0 > 0$ such that for all $\beta \in (0, \beta_0]$, the sequence $(\theta(n))$ obtained from Gq-SF1 converges to a point in the ϵ -neighborhood of the stable attractor of (39), defined as*

$$K^{\epsilon} = \{x : \|x - x_0\| < \epsilon, x_0 \in K\}$$

with probability 1 as $n \rightarrow \infty$.

Proof. It immediately follows from Lemmas 4.6 and 4.7 that the update in (33) converges to the stable fixed points of the ODE

$$\dot{\theta}(t) = \tilde{\mathcal{P}}_C \left(-\nabla_{\theta} J(\theta(t)) + \Delta(\theta(t)) \right). \quad (41)$$

Now starting from the same initial condition, the trajectory of (41) converges to that of (39) uniformly over compacts, as $\Delta(\theta(t)) \rightarrow 0$. Since from Proposition 4.8, we have $\|\Delta(\theta(n))\| = o(\beta)$ for all n , the claim follows. It may be noted that we can arrive at the same claim more technically using Hirsch's lemma (Hirsch, 1989). \square

4.2 Convergence of Gq-SF2 Algorithm

Since the proof of convergence here is along the lines of Gq-SF1, we do not describe it explicitly. We just briefly describe the modifications that are required in this case. In the faster timescale, as $n \rightarrow \infty$, the updates given by $Z(nL)$ track the function

$$\left(\frac{\eta(n)}{\beta(N + 2 - Nq)\rho(\eta(n))} \right) (J(\theta(n) + \beta\eta(n)) - J(\theta(n) - \beta\eta(n))).$$

So we can rewrite the slower timescale update for Gq-SF2 algorithm, in a similar manner as (33), where the noise term ξ_n has two components, due to the two parallel simulations, each being bounded (as in Lemma 4.7). We have the following proposition for the error term

$$\Delta(\theta(n)) = \nabla_{\theta} S'_{q,\beta}[J(\theta)] - \nabla_{\theta} J(\theta).$$

Proposition 4.10. For a given $q < (1 + \frac{2}{N})$, $q \neq 1$, and for all $\theta \in C$,

$$\left\| \nabla_{\theta} S'_{q,\beta}[J(\theta)] - \nabla_{\theta} J(\theta) \right\| = o(\beta).$$

Proof. Using Taylor's expansion, we have for small β ,

$$J(\theta + \beta\eta) - J(\theta - \beta\eta) = 2\beta\eta^T \nabla_{\theta} J(\theta) + o(\beta^2).$$

One can use similar arguments as in Proposition 4.8 to rewrite (22) as

$$\nabla_{\theta} S'_{q,\beta}[J(\theta)] = \frac{1}{(N + 2 - Nq)} \mathbb{E}_{G_q(\eta)} \left[\frac{2}{\rho(\eta)} \eta \eta^T \right] \nabla_{\theta} J(\theta) + o(\beta),$$

which leads to the claim. \square

Finally, we have a similar result to prove the convergence of the Gq-SF2 algorithm.

Theorem 4.11. Under Assumptions I – IV, for $\epsilon > 0$ and $q \in (-\infty, 1) \cup (1, 1 + \frac{2}{N})$, there exists $\beta_0 > 0$ such that for all $\beta \in (0, \beta_0]$, the sequence $(\theta(n))$ obtained from Gq-SF2 converges to a point in the ϵ -neighborhood of the stable attractor of (39), with probability 1 as $n \rightarrow \infty$.

Theorems 4.9 and 4.11 give the existence of some $\beta_0 > 0$ for a given $\epsilon > 0$ such that the gradient-descent algorithms converge to ϵ -neighborhood of a local minimum. However, these results do not give the precise value of β_0 . Further, they do not guarantee that this neighborhood lies within a close proximity of a global minimum.

We make a note on the analysis for Gaussian SF algorithms. Though the above results exclude the case $q = 1$, it is easy to verify that all the claims hold as $q \downarrow 1$ due to Corollary 4.2. Hence, the above convergence analysis provides an alternative to the analysis presented in (Bhatnagar, 2007) for Gaussian SF algorithms.

5 Simulations using the Proposed Algorithms

5.1 Numerical Setting

We consider a multi-node network of $M/G/1$ queues with feedback as shown in the figure below. There are K nodes, which are fed with independent Poisson external arrival processes with rates $\lambda_1, \lambda_2, \dots, \lambda_K$, respectively. After departing from the i^{th} node, a customer either leaves the system with probability p_i or enters the $(i + 1)^{th}$ node with probability $(1 - p_i)$. Once the service at the K^{th} node is completed, the customer may rejoin the 1^{st} node with probability $(1 - p_K)$. The service time processes of each node, $\{S_n^i(\theta_i)\}_{n \geq 1, i = 1, 2, \dots, K}$ are defined as

$$S_n^i(\theta_i) = U_i(n) \left(\frac{1}{R_i} + \|\theta_i(n) - \bar{\theta}_i\|^2 \right), \quad (42)$$

where for all $i = 1, 2, \dots, K$, R_i are constants and $U_i(n)$ are independent samples drawn from the uniform distribution on $(0, 1)$. The service time of each node depends on the N_i -dimensional tunable parameter vector θ_i , whose individual components lie in a certain interval $\left[\left(\theta_i^{(j)} \right)_{min}, \left(\theta_i^{(j)} \right)_{max} \right]$, $j = 1, 2, \dots, N_i$, $i = 1, 2, \dots, K$. $\theta_i(n)$ represents the n^{th} update of the parameter vector at the i^{th} node, and $\bar{\theta}_i$ represents the target parameter vector corresponding to the i^{th} node.

The cost function is chosen to be the sum of the total waiting times of all the customers in the system. For the cost to be minimum, $S_n^i(\theta_i)$ should be minimum, and hence, we should have $\theta_i(n) = \bar{\theta}_i$,

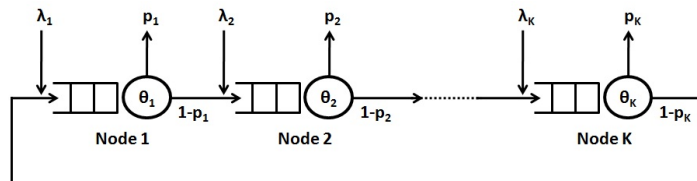


Figure 1: Queueing Network.

$i = 1, \dots, K$. Let us denote $\theta = (\theta_1, \theta_2, \dots, \theta_K)^T$ and $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2, \dots, \bar{\theta}_K)^T$. It is evident that $\theta, \bar{\theta} \in \mathbb{R}^N$, where $N = \sum_{i=1}^K N_i$. In order to compare the performance of the various algorithms, we consider the performance measure to be the Euclidean distance between $\theta(n)$ and $\bar{\theta}$,

$$\|\theta(n) - \bar{\theta}\| = \left[\sum_{i=1}^K \sum_{j=1}^{N_i} \left(\theta_i^{(j)}(n) - \bar{\theta}_i^{(j)} \right)^2 \right]^{1/2}.$$

The choice for such a performance measure is due to the fact that when the above distance is low, the queueing network provides globally optimal performance. Hence, in the results presented below, a low value of the distance (performance measure) implies that the algorithm converges to a closer proximity of the global minimum.

5.2 Experimental Results

For the simulations, we first consider a two queue network with the arrival rates at the nodes being $\lambda_1 = 0.2$ and $\lambda_2 = 0.1$ respectively. We consider that all customers leaving node-1 enter node-2, *i.e.*, $p_1 = 0$, while customers serviced at node-2 may leave the system with probability $p_2 = 0.4$. We also fix the constants in the service times at $R_1 = 10$ and $R_2 = 20$, respectively, for the two nodes. The service time parameters for either node are two-dimensional vectors, $N_1 = N_2 = 2$, with components lying in the interval $\left[\left(\theta_i^{(j)} \right)_{\min}, \left(\theta_i^{(j)} \right)_{\max} \right] = [0.1, 0.6]$ for $i = 1, 2, j = 1, 2$. Thus the constrained space C is given by $C = [0.1, 0.6]^4 \subset \mathbb{R}^4$. We fix the target parameter at $\bar{\theta} = (0.3, 0.3, 0.3, 0.3)^T$.

The simulations were performed on an Intel Core i5 machine with 3.7GiB memory space and Linux operating system. We run the algorithms by varying the values of q and β , while all the other parameters are held fixed at $M = 10000$ and $L = 100$. For all the cases, the initial parameter is assumed to be $\theta(0) = (0.1, 0.1, 0.6, 0.6)^T$. For each set of parameters, 20 independent runs were performed with each run of 10^6 iterations taking about 0.5 seconds. We compare the performance of the proposed algorithms with gradient based SF algorithms proposed in (Bhatnagar, 2007), which use Gaussian smoothing. Box-Müller method has been used to sample standard Gaussian vectors, while samples from multivariate q -Gaussians are drawn using the method discussed in Appendix. This method uses generation of a χ^2 -distributed random variable, which is implemented using standard methods described in (Kroese et al., 2011, Algorithms 4.33 and 4.37). Figure 2a shows the convergence behavior of the Gaussian and proposed q -Gaussian based algorithms with $q = 0.8$, where the smoothness parameter is $\beta = 0.005$.

Before going into a detailed study on the effect of q and smoothing parameter (β), we briefly discuss the effect of the step-sizes. In the plots shown in Figure 2a, the step-sizes $(a(n))_{n \geq 0}, (b(n))_{n \geq 0}$ were taken to be $a(n) = \frac{1}{n}, b(n) = \frac{1}{n^{0.75}}$, respectively, but these can be varied to control the rate of convergence of the gradient updates. We briefly address this in Table 1. We fix the step-size sequence $a(n) = \frac{1}{n}$, and vary the step-size $(b(n))$ for gradient estimation considering it to be of the form $b(n) = \frac{1}{n^\gamma}$. In order to satisfy Assumption IV, we need to consider $\gamma \in (0.5, 1)$. We study the performance of G q -SF1 and G q -SF2 with different values of γ . The value of $\beta = 0.005$ is chosen to be in the range where the Gaussian SF algorithms perform well (Bhatnagar, 2007), whereas the values of q are chosen over the range $(-\infty, 1 + \frac{2}{N}) = (-\infty, 1.5)$. It may be noted that $q \rightarrow 1$ corresponds to the Gaussian case, $q = 1.4$ corresponds to Cauchy, and $q \rightarrow -\infty$ gives the uniform distribution.

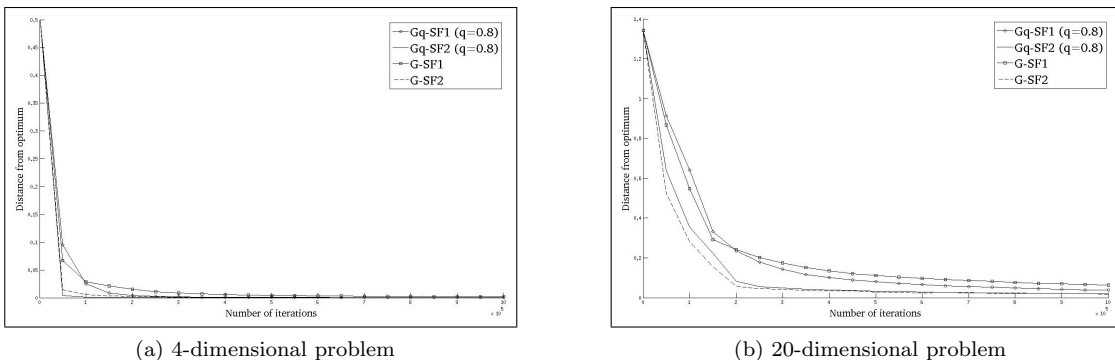


Figure 2: Convergence behaviour of SF and q -SF algorithms for $q = 0.8, \beta = 0.005$.

q	$\gamma = 0.65$		$\gamma = 0.75$		$\gamma = 0.85$	
	Gq-SF1	Gq-SF2	Gq-SF1	Gq-SF2	Gq-SF1	Gq-SF2
-10	0.06763±0.04668	0.03845±0.01814	0.06584±0.06456	0.06081±0.05528	0.06930±0.04607	0.04416±0.02763
-5	0.02340±0.02438	0.01123±0.00612	0.03341±0.04568	0.01225±0.00566	0.05830±0.06314	0.01559±0.01103
-1	0.00654±0.01030	0.00099±0.00065	0.00776±0.00877	0.00117±0.00080	0.00645±0.01470	0.00103±0.00091
-0.5	0.00184±0.00252	0.00055±0.00047	0.00248±0.00305	0.00035±0.00025	0.00299±0.00364	0.00032±0.00035
0	0.00058±0.00053	0.00026±0.00020	0.00258±0.00620	0.00035±0.00062	0.00110±0.00126	0.00016±0.00013
0.2	0.00122±0.00217	0.00013±0.00010	0.00087±0.00106	0.00011±0.00005	0.00148±0.00190	0.00015±0.00015
0.4	0.00056±0.00032	0.00012±0.00006	0.00064±0.00048	0.00012±0.00005	0.00110±0.00186	0.00012±0.00003
0.6	0.00041±0.00015	0.00011±0.00004	0.00050±0.00022	0.00011±0.00003	0.00057±0.00031	0.00012±0.00003
0.8	0.00051±0.00019	0.00011±0.00005	0.00124±0.00353	0.00013±0.00002	0.00049±0.00015	0.00014±0.00005
1.0 (Gaussian)	0.00336±0.00574	0.00056±0.00061	0.00259±0.00435	0.00030±0.00013	0.00128±0.00159	0.00052±0.00071
1.1	0.00048±0.00023	0.00014±0.00005	0.00049±0.00020	0.00014±0.00003	0.00067±0.00045	0.00013±0.00005
1.2	0.00060±0.00038	0.00014±0.00005	0.00052±0.00030	0.00016±0.00003	0.00081±0.00061	0.00016±0.00015
1.3	0.00064±0.00037	0.00014±0.00008	0.00055±0.00024	0.00018±0.00004	0.00115±0.00105	0.00014±0.00006
1.4 (Cauchy)	0.00164±0.00105	0.00090±0.00052	0.00157±0.00105	0.00062±0.00030	0.00155±0.00090	0.00101±0.00065
1.49	0.07281±0.05282	0.03600±0.00989	0.08478±0.04600	0.00494±0.00228	0.08474±0.04033	0.05174±0.01978

Table 1: Performance of all algorithm for different values of q with $\beta = 0.005$, and step-sizes $a(n) = \frac{1}{n}$ and $b(n) = \frac{1}{n^\gamma}$.

We show the distance of the final updates $\theta(M)$ from the optimum $\bar{\theta}$, averaged over 20 trials. The variance of the updates is also shown to indicate the robustness of the algorithms. The results agree with previous observations (Bhatnagar, 2007) that two-simulation algorithms perform better than their one-simulation counterparts. It also shows that in quite a number of cases, q -SF algorithms perform better than the Gaussian case. The results for different time-scales, controlled by γ , seem to be comparable. So we fix $\gamma = 0.75$ for further analysis.

We now focus on the effect of the value of q and the smoothing parameter β used in the algorithms. The results for Gq-SF1 and Gq-SF2 algorithms are presented in Tables 2 and 3. For each value of β , the instances where the performance for the q -Gaussian is better than Gaussian-SF are marked in bold. Also, the least distance obtained for each β is underlined. Similar trends in performance are observed in both the tables.

It can be seen that better results are usually obtained for $q > 1$ for low values of β . As β increases lower values of q give better performance, while the higher q -values result in higher variance. The extreme right column indicates a deterioration in the standard error performance, which can be attributed to the dependence of the error term on the parameter β . However, results for $q < 1$ seem to be less affected and hence, relative performance of algorithms with $q < 1$ are better for higher values of β . On the other hand, the noise term studied in Lemma 4.7 has two effects, observed in the left column and the bottom row of either tables. In the lemma, we obtain an upper bound on the variance of the noise ξ_n , which, though finite can be arbitrarily large for small β or q close to $(1 + \frac{2}{N})$. Thus, the result worsens in these cases. In fact, some simulations (not presented here) showed that the performance deteriorated further when β was made even smaller. The poor performance for small values of q , such as $q = -10$, can be argued in a very simple way. In such a case, the support of the distribution is a very small region around its mean at zero. Hence, the perturbations η are not large enough to achieve sufficient exploration that

$q \backslash \beta$	0.0005	0.001	0.005	0.01	0.05	0.1
-10	0.10301±0.09891	0.09882±0.09494	0.06584±0.06456	0.03360±0.02976	0.02857±0.01049	0.04079±0.01762
-5	0.11408±0.10771	0.09081±0.10812	0.03341±0.04568	0.01415±0.00970	0.01003±0.00596	0.01510±0.00622
-1	0.11220±0.13957	0.06315±0.10758	0.00776±0.00877	0.00183±0.00108	0.00367±0.00108	0.00646±0.00244
-0.5	0.06760±0.08813	0.06767±0.10344	0.00248±0.00305	0.00170±0.00175	0.00341±0.00103	0.00837±0.00204
0	0.11413±0.12973	0.03593±0.06865	0.00258±0.00620	0.00084±0.00033	0.00374±0.00142	0.00702±0.00247
0.2	0.10229±0.13098	0.03660±0.07375	0.00087±0.00106	0.00078±0.00028	0.00442±0.00125	0.00775±0.00331
0.4	0.05427±0.08606	0.00998±0.01887	0.00064±0.00048	0.00070±0.00033	0.00383±0.00137	0.00930±0.00313
0.6	0.05183±0.05849	0.02071±0.04122	0.00050±0.00022	0.00108±0.00097	0.00396±0.00163	0.00949±0.00343
0.8	0.11910±0.15236	0.03398±0.07965	0.00124±0.00353	0.00083±0.00032	0.00439±0.00114	0.01657±0.00433
Gaussian	0.05075±0.08472	0.02530±0.04332	0.00259±0.00435	0.00098±0.00047	0.00307±0.00109	0.01295±0.00337
1.1	0.05020±0.09131	0.00858±0.02574	0.00049±0.00020	0.00091±0.00030	0.00556±0.00220	0.04563±0.00631
1.2	0.01455±0.02134	0.00402±0.00937	0.00052±0.00030	0.00089±0.00030	0.00577±0.00133	0.04694±0.00663
1.3	0.04144±0.06542	0.01225±0.02636	0.00055±0.00024	0.00099±0.00034	0.00735±0.00236	0.05034±0.00572
Cauchy	0.02982±0.04785	0.00575±0.00671	0.00157±0.00105	0.00237±0.00075	0.01602±0.00319	0.06032±0.00651
1.49	0.24013±0.10511	0.17490±0.07159	0.08478±0.04600	0.04875±0.02921	0.05422±0.01747	0.09141±0.03698

Table 2: Performance of Gq-SF1 algorithm for different values of q and β .

$q \backslash \beta$	0.0005	0.001	0.005	0.01	0.05	0.1
-10	0.06036±0.04502	0.08081±0.05699	0.06081±0.05528	0.02630±0.00787	0.02245±0.00741	0.02176±0.00894
-5	0.09589±0.11057	0.02078±0.02135	0.01225±0.00566	0.00650±0.00407	0.00527±0.00260	0.00819±0.00448
-1	0.01731±0.03489	0.02113±0.02057	0.00117±0.00080	0.00040±0.00015	0.00100±0.00022	0.00216±0.00064
-0.5	0.05658±0.10012	0.00362±0.00572	0.00035±0.00025	0.00019±0.00008	0.00104±0.00027	0.00165±0.00050
0	0.00336±0.00264	0.00136±0.00256	0.00035±0.00062	0.00019±0.00010	0.00090±0.00032	0.00214±0.00070
0.2	0.00481±0.00475	0.00090±0.00086	0.00011±0.00005	0.00016±0.00008	0.00103±0.00048	0.00314±0.00124
0.4	0.00473±0.00421	0.00076±0.00103	0.00012±0.00005	0.00026±0.00007	0.00091±0.00021	0.00327±0.00078
0.6	0.00581±0.00276	0.00062±0.00113	0.00011±0.00003	0.00025±0.00010	0.00112±0.00061	0.00618±0.00149
0.8	0.00100±0.00194	0.00026±0.00037	0.00013±0.00002	0.00023±0.00007	0.00142±0.00061	0.01331±0.00146
Gaussian	0.00483±0.00531	0.00160±0.00142	0.00030±0.00013	0.00026±0.00015	0.00094±0.00044	0.01119±0.00168
1.1	0.00102±0.00178	0.00033±0.00048	0.00014±0.00003	0.00026±0.00012	0.00139±0.00045	0.04619±0.00229
1.2	0.00120±0.00154	0.00018±0.00019	0.00016±0.00003	0.00022±0.00008	0.00208±0.00035	0.04432±0.00286
1.3	0.00316±0.00670	0.00074±0.00061	0.00018±0.00004	0.00030±0.00014	0.00515±0.00066	0.04825±0.00176
Cauchy	0.00314±0.00398	0.00145±0.00135	0.00062±0.00030	0.00138±0.00053	0.01462±0.00060	0.05675±0.00373
1.49	0.00424±0.00226	0.00600±0.00431	0.00494±0.00228	0.00682±0.00230	0.02771±0.00178	0.06799±0.00536

Table 3: Performance of Gq-SF2 algorithm for different values of q and β .

would be required to reach a good optimum.

Another interesting phenomena is observed that requires some discussion. For β in the range 0.001 to 0.01, it is seen that the results for the Gaussian case ($q \rightarrow 1$) are poor when compared with those for q close to 1. This result appears counter-intuitive as the performance of the Gaussian SF “should have been” a limiting case of the q -Gaussian SF. Hence, this observation needs a detailed analysis. A closer look at the presented algorithms shows that the value of q plays a role only in two steps - generation of the perturbations η , and the gradient update rule. It is easy to verify that the gradient update is a continuous function of q at $q = 1$, and hence, cannot induce the observed effect. Thus, the issue lies with the sampling step, which is in turn dependent on the χ^2 variate used in the sampling method discussed in Appendix. One can verify that for q close to 1, the parameter of the χ^2 random variable, say m (used in Step 2 of sampling method), is a large positive quantity, and so implementations involve Algorithm 4.33 from (Kroese et al., 2011). Assuming m is quite large, it follows from Algorithm 4.33 that that random variate $a \sim \chi^2(m)$ is close to $\frac{m}{2}$, and so, roughly $Y \approx \sqrt{2}Z$. On the other hand, for the case of $q = 1$ (Gaussian), $Y = Z$. Hence, the perturbations are larger for q close to 1 as compared to $q = 1$, which helps the algorithms to avoid getting stuck at any local minimum. This phenomena is not observed as β increases due to increased smoothing. Here, relative performance of Gaussian SF is better, but in these cases, the overall performance of the algorithms becomes worse and hence, no benefit is achieved by the Gaussian.

We perform similar experiments in a higher dimensional case. For this, we consider a four node network with $\lambda_i = 0.2$ for all $i = 1, \dots, 4$. The probability of leaving the system after service at each node is $p_i = 0.2$ for all nodes. The service process of each node is controlled by a 5-dimensional parameter vector, and a constant set at $R_i = 10$. Thus, we have a 20-dimensional constrained optimization problem, where each component can vary over the interval $[0.1, 0.6]$ and the target is 0.3. The parameters of the

$q \backslash \beta$	0.005	0.0075	0.01	0.025	0.05	0.075	0.1
-10	0.4510±0.0877	0.3617±0.1100	0.2749±0.0534	0.2051±0.0246	0.1687±0.0240	0.1651±0.0205	0.1758±0.0316
-5	0.4072±0.1072	0.3079±0.0776	0.2635±0.0572	0.1236±0.0221	0.0972±0.0176	0.0787±0.0140	0.0961±0.0249
-1	0.2629±0.1003	0.1534±0.0569	0.1392±0.0877	0.0368±0.0108	0.0199±0.0047	0.0196±0.0024	0.0320±0.0068
-0.5	0.3852±0.1728	0.1265±0.0509	0.0820±0.0278	0.0277±0.0088	0.0146±0.0039	0.0136±0.0031	0.0297±0.0037
0	0.2718±0.1072	0.1186±0.0423	0.0742±0.0357	0.0137±0.0035	0.0099±0.0024	0.0134±0.0011	0.0282±0.0036
0.2	0.2580±0.1134	0.1393±0.0709	0.0575±0.0192	0.0136±0.0068	0.0087±0.0017	0.0130±0.0018	0.0271±0.0047
0.4	0.2271±0.1073	0.0875±0.0344	0.0649±0.0202	0.0086±0.0024	0.0090±0.0016	0.0122±0.0018	0.0300±0.0049
0.6	0.2643±0.1757	0.0746±0.0331	0.0397±0.0289	0.0081±0.0028	0.0085±0.0018	0.0124±0.0019	0.0298±0.0031
0.8	0.1627±0.0593	0.0706±0.0307	0.0256±0.0109	0.0056±0.0012	0.0086±0.0014	0.0139±0.0023	0.0344±0.0039
Gaussian	0.2042±0.0525	0.0946±0.0331	0.0770±0.0156	0.0242±0.0056	0.0143±0.0026	0.0153±0.0016	0.0308±0.0032
1.02	0.0850±0.0393	0.0360±0.0164	0.0169±0.0093	0.0055±0.0010	0.0116±0.0023	0.0373±0.0043	0.1150±0.0073
1.04	0.0651±0.0226	0.0407±0.0187	0.0255±0.0087	0.0074±0.0016	0.0118±0.0013	0.0368±0.0045	0.1140±0.0063
1.06	0.1055±0.0577	0.0364±0.0103	0.0281±0.0122	0.0089±0.0018	0.0122±0.0025	0.0408±0.0045	0.1208±0.0086
1.08	0.0858±0.0419	0.0522±0.0202	0.0329±0.0124	0.0136±0.0055	0.0157±0.0022	0.0542±0.0069	0.1405±0.0104
Cauchy	0.1153±0.0445	0.0701±0.0237	0.0572±0.0365	0.0292±0.0070	0.0578±0.0051	0.1118±0.0122	0.1980±0.0134
1.099	0.4316±0.0794	0.3314±0.0630	0.2787±0.0460	0.1916±0.0284	0.1836±0.0420	0.2268±0.0337	0.2781±0.0342

Table 4: Performance of the Gq-SF2 algorithm for different values of q and β .

algorithms are held fixed at $M = 10000$, $L = 100$ and $\varepsilon = 0.1$. Each component in the initial parameter vector is assumed to be $\theta^{(i)}(0) = 0.6$ for all $i = 1, 2, \dots, 20$. The step-sizes were taken to be $a(n) = \frac{1}{n}$ and $b(n) = \frac{1}{n^{0.85}}$. For each (q, β) tuple, 20 independent runs were performed. Each run of 10^6 iterations took about 1 second on an average. Figure 2b shows the convergence behavior of the algorithms in this case. The performance of one-simulation algorithm is relatively poor. Table 4 shows the performance of Gq-SF2 algorithms for different (q, β) -pairs. We observe that the trend is similar in this case showing that the algorithms scale in a similar manner.

6 Conclusions

The q -Gaussian distribution is an important power-law distribution that has connections with generalized information measures. The power-law behavior of q -Gaussians provide a better control over smoothing of functions as compared to the Gaussian distribution. We have extended the Gaussian smoothed functional approach for gradient estimation to the q -Gaussian case by showing that the q -Gaussian distribution satisfies the Rubinstein conditions (Rubinstein, 1981). Further, we developed optimization algorithms that incorporate q -Gaussian smoothing. This extension turns out to be more significant when we note that the q -Gaussians encompass all the existing smoothing kernels - Gaussian ($q \rightarrow 1$), Cauchy ($q = 1 + \frac{2}{N+1}$) and uniform ($q \rightarrow -\infty$).

We proposed two q -Gaussian SF algorithms for simulation optimization. We use a queuing network example to show that for certain values of q , the results provided by the proposed algorithms are significantly better than Gaussian SF algorithms. Our simulation results even indicate that for some q -values, the performance is better as compared to all the above mentioned special cases. We also presented proof of convergence of the proposed algorithms to a local minimum of the objective function.

It would be interesting to develop Hessian estimators incorporating the q -Gaussian smoothed functionals, and developing Newton based algorithms along these lines.

APPENDIX

Sampling algorithm for multivariate q -Gaussian distribution

The algorithms discussed in the paper require generation of a multivariate q -Gaussian distributed random vector, whose individual components are uncorrelated and identically distributed. This implies that the random variables are q -independent (Umarov and Tsallis, 2007). For the limiting case of $q \rightarrow 1$, q -independence is equivalent to independence of the random variables. Hence, we can use standard algorithms to generate i.i.d. samples. This is typically not possible for q -Gaussians with $q \neq 1$. Thistleton et al. (2007) proposed an algorithm for generating one-dimensional q -Gaussian distributed random variables using generalized Box-Müller transformation. But, there exists no standard algorithm for generating N -variate q -Gaussian random vectors.

A method can be obtained by making use of the one-to-one correspondence between q -Gaussian and Students'- t distributions for $q > 1$. Further, a duality property of q -Gaussians can be used to relate the distributions for $q \in (1, 1 + \frac{2}{N+2})$ and $q \in (-\infty, 1)$. This observation, first made by Vignat and Plastino (2006), is shown below. We denote the q -Gaussian distribution with q -mean μ_q and q -covariance Σ_q as $\mathcal{G}_q(\mu_q, \Sigma_q)$. Based on this, we formally present an algorithm, which can be used to generate multivariate q -Gaussian distribution. Theoretical justification behind the algorithm is provided in the following results (Vignat and Plastino, 2006).

Lemma A.1. *Given $Z \sim \mathcal{N}(0, I_{N \times N})$ and $a \sim \chi^2(m)$, $m > 0$, then the vector $Y = \sqrt{\frac{m}{a}}Z \sim \mathcal{G}_q(0, I_{N \times N})$, where $q = \left(1 + \frac{2}{N+m}\right)$.*

Lemma A.2. *Let $Y \sim \mathcal{G}_q(0, I_{N \times N})$ for some $q \in (1, 1 + \frac{2}{N+2})$ and*

$$X = \frac{\sqrt{\frac{2-q}{N+2-Nq}}Y}{\sqrt{1 + \frac{q-1}{N+2-Nq}Y^TY}}.$$

Then $X \sim \mathcal{G}_{q'}(0, I_{N \times N})$, where $q' = \left(1 - \frac{q-1}{(N+4)-(N+2)q}\right)$.

Lemma A.3. If $Y \sim \mathcal{G}_q(0, I_{N \times N})$ for some $q \in (-\infty, 1) \cup (1, 1 + \frac{2}{N})$, then

$$X = (\mu_q + \Sigma_q^{1/2} Y) \sim \mathcal{G}_q(\mu_q, \Sigma_q).$$

Sampling algorithm for multivariate q -Gaussian distribution

Input:

- a) $q \in (-\infty, 1 + \frac{2}{N})$, where $q = 1$ is assumed to be the Gaussian case
 - b) q -mean, $\mu_q \in \mathbb{R}^N$
 - c) q -covariance matrix $\Sigma_q \in \mathbb{R}^{N \times N}$
- 1 Generate N -dimensional standard Gaussian vector $Z \sim \mathcal{N}(0, I_{N \times N})$.
 - 2 Generate chi-squared random variate (Kroese et al., 2011, Chapter 4.2.6)

$$a \sim \begin{cases} \chi^2 \left(\frac{2(2-q)}{1-q} \right) & \text{for } -\infty < q < 1, \\ \chi^2 \left(\frac{N+2-Nq}{q-1} \right) & \text{for } 1 < q < (1 + \frac{2}{N}). \end{cases}$$

- 3 Compute

$$Y = \begin{cases} \sqrt{\frac{N+2-Nq}{1-q}} \frac{Z}{\sqrt{a+Z^T Z}} & \text{for } -\infty < q < 1, \\ Z & \text{for } q = 1, \\ \sqrt{\frac{N+2-Nq}{q-1}} \frac{Z}{\sqrt{a}} & \text{for } 1 < q < (1 + \frac{2}{N}). \end{cases}$$

Output: $X = (\mu_q + \Sigma_q^{1/2} Y)$, which is a sample from $\mathcal{G}_q(\mu_q, \Sigma_q)$.

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